

Zooming from Global to Local: A Multiscale RBF Approach

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Abstract

Because physical phenomena on Earth's surface occur on many different length scales, it makes sense when seeking an efficient approximation to start with a crude global approximation, and then make a sequence of corrections on finer and finer scales. It also makes sense eventually to seek fine scale features locally, rather than globally. In the present work, we start with a global multiscale radial basis function (RBF) approximation, based on a sequence of point sets with decreasing mesh norm, and a sequence of (spherical) radial basis functions with proportionally decreasing scale centered at the points. We then prove that we can “zoom in” on a region of particular interest, by carrying out further stages of multiscale refinement on a local region. The proof combines multiscale techniques for the sphere from Le Gia, Sloan and Wendland, SIAM J. Numer. Anal. 48 (2010) and Applied Comp. Harm. Anal. 32 (2012), with those for a bounded region in \mathbb{R}^d from Wendland, Numer. Math. 116 (2012). The zooming in process can be continued indefinitely, since the condition numbers of matrices at the different scales remain bounded. A numerical example illustrates the process.

1 Introduction

In many modern areas of geosciences, such as geomagnetic or gravitational field modeling, the problem of interpolation from scattered data on the sphere arises naturally. Such problems are often of multiscale nature, so one would like models that can be used to draw conclusions globally as well as locally. For example, in modeling the global gravitational field one should be able to see the general nature of the global field as well as local gravitational anomalies.

Multiscale interpolation and approximation for functions on the unit sphere has been considered by a number of authors using different techniques. Some authors used wavelets defined on spheres [2, 24], but these are not suitable for scattered data. Other authors have proposed kernel methods based on truncations of the expansions of some special kernels into spherical harmonics [5, 18, 15]; these methods require a quadrature scheme on the sphere which can integrate spherical polynomials exactly,

but the construction of a good quadrature based on scattered data is itself a non-trivial problem [6, 8, 11, 4, 16].

In recent articles [12, 13], we proposed a multiscale interpolation framework using radial basis functions for functions that lie in Sobolev spaces defined on the unit sphere. The theory underlying our multiscale method will work for scattered data. In this paper we introduce a “zooming-in” framework, which allows the multiscale algorithm to model the data from the global scale and then zoom in to local regions. We do this by combining multiscale techniques for the sphere with those for a bounded region established in [27].

The paper is organised as follows. In Section 2 we review necessary materials about Sobolev spaces on spheres and positive definite kernels defined via radial basis functions (RBFs). The global and local multiscale algorithm using spherical RBFs is then introduced in Section 3. A convergence results for functions in the native space is given there. The next section, Section 4, deals with convergence results for function in Sobolev spaces with lesser smoothness. Finally we conclude the paper with numerical experiments given in Section 5.

2 Preliminaries

In this section, we will introduce necessary materials for the main results presented in the paper.

2.1 Sobolev spaces on the unit sphere

Let \mathbb{S}^d be the unit sphere in \mathbb{R}^{d+1} . Denote the inner product in $L_2(\mathbb{S}^d)$ by

$$\langle v, w \rangle := \int_{\mathbb{S}^d} v w dS,$$

where dS is the surface measure on the unit sphere, and denote the measure of the whole sphere by ω_d (so, for example, $\omega_2 = 4\pi$). Recall [17] that a spherical harmonic is the restriction to \mathbb{S}^d of a homogeneous harmonic polynomial $Y(\mathbf{x})$ in \mathbb{R}^{d+1} . The space of all spherical harmonics of degree ℓ on \mathbb{S}^d , denoted by H_ℓ , has an L_2 orthonormal basis

$$\{Y_{\ell,k} : k = 1, \dots, N(d, \ell)\},$$

where

$$N(d, 0) = 1 \text{ and } N(d, \ell) = \frac{(2\ell + d - 1)\Gamma(\ell + d - 1)}{\Gamma(\ell + 1)\Gamma(d)} \text{ for } \ell \geq 1.$$

The space of spherical harmonics of degree $\leq L$ will be denoted by $\mathcal{P}_L := \oplus_{\ell=0}^L H_\ell$; it has dimension $N(d+1, L)$.

Every function $f \in L_2(\mathbb{S}^d)$ can be expanded in terms of spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \hat{f}_{\ell,k} Y_{\ell,k}, \quad \hat{f}_{\ell,k} = \langle f, Y_{\ell,k} \rangle.$$

For a non-negative parameter σ , the Sobolev space $H^\sigma(\mathbb{S}^d)$ may be defined by

$$H^\sigma(\mathbb{S}^d) := \left\{ f \in L_2(\mathbb{S}^d) : \|f\|_{H^\sigma(\mathbb{S}^d)}^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} (1+\ell)^{2\sigma} |\widehat{f}_{\ell,k}|^2 < \infty \right\}. \quad (1)$$

Note that $H^0(\mathbb{S}^d) = L_2(\mathbb{S}^d)$.

Sobolev spaces on \mathbb{S}^d can also be defined using local charts (see [14]). Here we use a specific atlas of charts, as in [9].

Let \mathbf{z} be a given point on \mathbb{S}^d . The spherical cap centered at \mathbf{z} of radius θ is defined by

$$G(\mathbf{z}, \theta) := \{ \mathbf{y} \in \mathbb{S}^d : \cos^{-1}(\mathbf{z} \cdot \mathbf{y}) < \theta \}, \quad \theta \in (0, \pi),$$

where $\mathbf{z} \cdot \mathbf{y}$ denotes the Euclidean inner product of \mathbf{z} and \mathbf{y} in \mathbb{R}^{d+1} .

Let $\hat{\mathbf{n}}$ and $\hat{\mathbf{s}}$ denote the north and south poles of \mathbb{S}^d , respectively. Then a simple cover for the sphere is provided by

$$U_1 = G(\hat{\mathbf{n}}, \theta_0) \quad \text{and} \quad U_2 = G(\hat{\mathbf{s}}, \theta_0), \quad \text{where } \theta_0 \in (\pi/2, 2\pi/3). \quad (2)$$

The stereographic projection $\sigma_{\hat{\mathbf{n}}}$ of the punctured sphere $\mathbb{S}^d \setminus \{\hat{\mathbf{n}}\}$ onto \mathbb{R}^d is defined as a mapping that maps $\mathbf{x} \in \mathbb{S}^d \setminus \{\hat{\mathbf{n}}\}$ to the intersection of the equatorial hyperplane $\{\mathbf{z} = 0\}$ and the extended line that passes through \mathbf{x} and $\hat{\mathbf{n}}$. The stereographic projection $\sigma_{\hat{\mathbf{s}}}$ based on $\hat{\mathbf{s}}$ can be defined analogously. We set

$$\psi_1 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\mathbf{s}}} |_{U_1} \quad \text{and} \quad \psi_2 = \frac{1}{\tan(\theta_0/2)} \sigma_{\hat{\mathbf{n}}} |_{U_2}, \quad (3)$$

so that ψ_k , $k = 1, 2$, maps U_k onto $B(0, 1)$, the unit ball in \mathbb{R}^d . We conclude that $\mathcal{A} = \{U_k, \psi_k\}_{k=1,2}$ is a C^∞ atlas of covering coordinate charts for the sphere. It is known (see [21]) that the stereographic coordinate charts $\{\psi_k\}_{k=1,2}$ as defined in (3) map spherical caps to Euclidean balls, but in general concentric spherical caps are not mapped to concentric Euclidean balls. The projection ψ_k , for $k = 1, 2$, does not distort too much the geodesic distance between two points $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$, as shown in [10].

With the atlas so defined, we define the map π_k which takes a real-valued function g with compact support in U_k into a real-valued function on \mathbb{R}^d by

$$\pi_k(g)(\mathbf{x}) = \begin{cases} g \circ \psi_k^{-1}(\mathbf{x}), & \text{if } \mathbf{x} \in B(0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{\chi_k : \mathbb{S}^d \rightarrow \mathbb{R}\}_{k=1,2}$ be a partition of unity subordinated to the atlas, i.e., a pair of non-negative infinitely differentiable functions χ_k on \mathbb{S}^d with compact support in U_k , such that $\sum_k \chi_k = 1$. For any function $f : \mathbb{S}^d \rightarrow \mathbb{R}$, we can use the partition of unity to write

$$f = \sum_{k=1}^2 \chi_k f, \quad \text{where } (\chi_k f)(\mathbf{x}) = \chi_k(\mathbf{x}) f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{S}^d.$$

The Sobolev space $H^\sigma(\mathbb{S}^d)$ is then the set

$$\left\{ f \in L_2(\mathbb{S}^d) : \pi_k(\chi_k f) \in H^\sigma(\mathbb{R}^d) \quad \text{for } k = 1, 2 \right\},$$

which is equipped with the norm

$$\|f\|_{H^\sigma(\mathbb{S}^d)} := \left(\sum_{k=1}^2 \|\pi_k(\chi_k f)\|_{H^\sigma(\mathbb{R}^d)}^2 \right)^{1/2}. \quad (4)$$

This norm is equivalent to the $H^\sigma(\mathbb{S}^d)$ norm given in (1) (see [14]). From now on we will use only the $\|\cdot\|$ notation for the equivalent norms.

We recall that [1] the Sobolev space $H^\sigma(\mathbb{R}^d)$ is the set

$$\left\{ f \in L_2(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\mathcal{F}(f)(\omega)|^2 (1 + \|\omega\|_2^2)^\sigma d\omega < \infty \right\},$$

where $\mathcal{F}(f)$ is the usual Fourier transform

$$\mathcal{F}(f)(\omega) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\mathbf{x}^T \omega} d\mathbf{x}.$$

Before introducing local Sobolev spaces on subdomains of the unit sphere, let us recall a few key definitions on Sobolev spaces defined on a given bounded domain D in \mathbb{R}^d . For a given non-negative integer m , the Sobolev space $H^m(D)$ consist of all f with weak derivatives $D^\alpha f \in L_2(D)$, $|\alpha| \leq m$. The semi-norms and norms are defined by

$$|f|_{H^m(D)} = \left(\sum_{|\alpha|=m} \|D^\alpha f\|_{L_2(D)}^2 \right)^{1/2} \quad \text{and} \quad \|f\|_{H^m(D)} = \left(\sum_{|\alpha| \leq m} \|D^\alpha f\|_{L_2(D)}^2 \right)^{1/2}.$$

For $m \in \mathbb{N}_0$, $0 < s < 1$, the fractional Sobolev spaces $H^{m+s}(D)$ is defined to be the set of all f for which the following semi-norm and norm

$$\begin{aligned} |f|_{H^{m+s}(D)} &:= \left(\sum_{|\alpha|=m} \int_D \int_D \frac{|D^\alpha f(\mathbf{x}) - D^\alpha f(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|_2^{d+2s}} \right)^{1/2} \\ \|f\|_{H^{m+s}(D)} &:= (\|f\|_{H^m(D)}^2 + |f|_{H^{m+s}(D)}^2)^{1/2} \end{aligned}$$

are finite.

Let $\Omega \subset \mathbb{S}^d$ be an open connected set with sufficiently smooth boundary. In order to define the spaces on Ω , let $D_k = \psi_k(\Omega \cap U_k)$ for $k = 1, 2$. The local Sobolev space $H^\sigma(\Omega)$ is defined to be the set

$$\{f \in L_2(\Omega) : \pi_k(\chi_k f)|_{D_k} \in H^\sigma(D_k) \text{ for } k = 1, 2, D_k \neq \emptyset\},$$

which is equipped with the norm

$$\|f\|_{H^\sigma(\Omega)} = \left(\sum_{k=1}^2 \|\pi_k(\chi_k f)|_{D_k}\|_{H^\sigma(D_k)}^2 \right)^{1/2} \quad (5)$$

where, if $\Omega = \emptyset$, then we adopt the convention that $\|\cdot\|_{H^\sigma(D_k)} = 0$.

We observe, following [9], that there exists a positive constant $C_{\mathcal{A}}$, depending on \mathcal{A} and the partition of unity $\{\chi_1, \chi_2\}$, such that the geodesic distance of $\text{supp } \chi_k$ from

the boundary of U_k is strictly greater than C_A . A spherical cap $G(\mathbf{z}, \theta)$ with $\theta < C_A/3$ will have its closure being a subset of at least one of the open subsets U_1 or U_2 , defined by (2), and if the cap $G(\mathbf{z}, \theta)$ is not a subset of one of these subsets, say U_2 , then its intersection with $\text{supp } \chi_2$ must be empty.

Now we state an extension theorem for a spherical cap on the sphere.

Theorem 2.1 (Extension operator) *Let $\Omega = G(\mathbf{z}, \theta)$ be a spherical cap for some $\mathbf{z} \in \mathbb{S}^d$ and $\theta < C_A/3$. There is an extension operator $E : H^\nu(\Omega) \rightarrow H^\nu(\mathbb{S}^d)$ for all $\nu \geq 0$, with E independent of ν , such that*

1. $Ef|_\Omega = f$ for all $f \in H^\nu(\Omega)$,
2. $\|Ef\|_{H^\nu(\mathbb{S}^d)} \leq C_\nu \|f\|_{H^\nu(\Omega)}$.

Proof: The case of ν being an integer was proved in [9, Theorem 4.3]. The framework for the case of fractional order ν is also available in [9] even if the explicit statement is not given there. For the sake of completeness, we give the proof here.

When ν is not an integer, let k be the non-negative integer for which $\nu = k + s$, with $s \in (0, 1)$. By [9, Theorem 4.3], there is an extension operator which maps $H^{k+i}(\Omega)$ to $H^{k+i}(\mathbb{S}^d)$ and there are constants $C_{k,i}$ for $i = 0, 1$ so that

$$\|Ef\|_{H^{k+i}(\mathbb{S}^d)} \leq C_{k,i} \|f\|_{H^{k+i}(\Omega)}, \quad i = 0, 1.$$

Using the *operator interpolation property* (see [25]) we conclude that E is a bounded linear map from $H^\nu(\Omega)$ to $H^\nu(\mathbb{S}^d)$ and

$$\|Ef\|_{H^\nu(\mathbb{S}^d)} \leq C_{k,0}^{1-s} C_{k,1}^s \|f\|_{H^\nu(\Omega)}.$$

Property 1) follows from the fact that $H^\nu(\mathbb{S}^d) \subset H^k(\mathbb{S}^d)$ and $H^\nu(\Omega) \subset H^k(\Omega)$. \square

2.2 Positive definite kernels on the unit sphere

A continuous function $\Phi : \mathbb{S}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ we call a *positive semi-definite kernel* [23, 28] on \mathbb{S}^d if it satisfies the following conditions:

- (i) Φ is continuous,
- (ii) $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$,
- (iii) For any set of distinct scattered points $X = \{\mathbf{x}_1, \dots, \mathbf{x}_K\} \subset \mathbb{S}^d$, the symmetric $K \times K$ -matrix $[\Phi(\mathbf{x}_p, \mathbf{x}_q)]$ is positive semi-definite.

We call Φ positive definite if the matrix is positive definite.

We will work with a zonal kernel Φ defined in terms of a univariate function $\phi : [-1, 1] \rightarrow \mathbb{R}$ by

$$\Phi(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} \cdot \mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{S}^d. \quad (6)$$

Following Müller [17], let $P_\ell(t)$ denote the Legendre polynomial of degree ℓ for \mathbb{R}^{d+1} , and expand $\phi(t)$ in a Fourier–Legendre series

$$\phi(t) = \frac{1}{\omega_d} \sum_{\ell=0}^{\infty} N(d, \ell) \widehat{\phi}(\ell) P_{\ell}(t). \quad (7)$$

Due to the addition formula for spherical harmonics [17, page 10]

$$\sum_{k=1}^{N(n, \ell)} Y_{\ell, k}(\mathbf{x}) Y_{\ell, k}(\mathbf{y}) = \frac{N(d, \ell)}{\omega_d} P_{\ell}(\mathbf{x} \cdot \mathbf{y}), \quad (8)$$

the kernel Φ can be represented as

$$\Phi(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{N(d, \ell)} \widehat{\phi}(\ell) Y_{\ell, k}(\mathbf{x}) Y_{\ell, k}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

and since $P_{\ell}(1) = 1$ we find that

$$\|\Phi(\mathbf{x}, \cdot)\|_{H^{\sigma}(\mathbb{S}^d)}^2 = \frac{1}{\omega_d} \sum_{\ell=0}^{\infty} (1 + \ell)^{2\sigma} (\widehat{\phi}(\ell))^2 N(d, \ell), \quad \text{for all } \mathbf{x} \in \mathbb{S}^d. \quad (9)$$

Chen et al. [3] proved that the kernel Φ is positive definite if and only if $\widehat{\phi}(\ell) \geq 0$ for all $\ell \geq 0$ and $\widehat{\phi}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ ; see also Schoenberg [23] and Xu and Cheney [28]. Here, we assume there is a $\sigma > d/2$ and there are positive constants c_1 and c_2 such that

$$c_1^2 (1 + \ell)^{-2\sigma} \leq \widehat{\phi}(\ell) \leq c_2^2 (1 + \ell)^{-2\sigma}, \quad \text{for all } \ell \geq 0, \quad (10)$$

hence, Φ is positive definite. Also, since $N(d, \ell) = O(\ell^{d-1})$ as $\ell \rightarrow \infty$, the sum (9) is finite for each fixed $\mathbf{x} \in \mathbb{S}^d$. Thus the function $\mathbf{y} \mapsto \Phi(\mathbf{x}, \mathbf{y})$ belongs to $H^{\sigma}(\mathbb{S}^d)$. Moreover, this function is continuous by the Sobolev imbedding theorem.

The reproducing kernel Hilbert space (RKHS) (also called the *native space*) induced by Φ is defined to be

$$\mathcal{N}_{\Phi} = \left\{ f \in L_2(\mathbb{S}^d) : \|f\|_{\Phi}^2 := \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d, \ell)} \frac{|\widehat{f}_{\ell, k}|^2}{\widehat{\phi}(\ell)} < \infty \right\}. \quad (11)$$

Alternatively, \mathcal{N}_{Φ} is the completion of $\text{span}\{\Phi(\cdot, \mathbf{x}) : \mathbf{x} \in \mathbb{S}^d\}$ with respect to the norm $\|\cdot\|_{\Phi}$. The norm is associated with the following inner product

$$(f, g)_{\Phi} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d, \ell)} \frac{\widehat{f}_{\ell, k} \widehat{g}_{\ell, k}}{\widehat{\phi}(\ell)}, \quad f, g \in \mathcal{N}_{\Phi}.$$

The kernel Φ has the reproducing property with respect to this inner product, that is

$$f(\mathbf{x}) = (f, \Phi(\cdot, \mathbf{x}))_{\Phi}, \quad \mathbf{x} \in \mathbb{S}^d, f \in \mathcal{N}_{\Phi}. \quad (12)$$

It follows from (10) that the norms in $H^{\sigma}(\mathbb{S}^d)$ and \mathcal{N}_{Φ} are equivalent.

2.3 Kernels defined from radial basis functions

Let $\Pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a compactly supported radial basis function (RBF) with associated RKHS $H^\tau(\mathbb{R}^{d+1})$ with $\tau > (d+1)/2$. Examples of such RBFs are the Wendland functions (see [26]).

By restricting the function Π to the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we have a positive definite, zonal kernel on the unit sphere

$$\Phi(\mathbf{x}, \mathbf{y}) = \Pi(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

Lemma 2.2 (Native spaces) *Let $\Pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a positive definite function with native space $\mathcal{N}_\Pi(\mathbb{R}^{d+1}) = H^\tau(\mathbb{R}^{d+1})$ with $\tau > (d+1)/2$. Then $\mathcal{N}_\Phi(\mathbb{S}^d) = H^\sigma(\mathbb{S}^d)$ with $\sigma = \tau - \frac{1}{2}$.*

Proof: Using [19, Proposition 4.2], we deduce that

$$c(1 + \ell)^{-2\sigma} \leq \widehat{\phi}(\ell) \leq C(1 + \ell)^{-2\sigma}.$$

So the result follows from the definition of the Sobolev spaces (1) and the native spaces (11) on \mathbb{S}^d . \square

For a given $\delta > 0$, we define the scaled version Φ_δ of the kernel Φ by

$$\Phi_\delta(\mathbf{x}, \mathbf{y}) = \delta^{-d} \Pi((\mathbf{x} - \mathbf{y})/\delta). \quad (13)$$

We can expand Φ_δ into a series of spherical harmonics

$$\Phi_\delta(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \widehat{\phi}_\delta(\ell) Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}),$$

in which the Fourier coefficients satisfy the following condition (see [12, Theorem 6.2])

$$c_1^2(1 + \delta\ell)^{-2\sigma} \leq \widehat{\phi}_\delta(\ell) \leq c_2^2(1 + \delta\ell)^{-2\sigma}, \quad (14)$$

with the coefficients c_1 and c_2 from (10) possibly relaxed so that (14) holds for all $0 < \delta \leq 1$.

For a function $f \in H^\sigma(\mathbb{S}^d)$, we define the norm corresponding to the scaled kernel Φ_δ by

$$\|f\|_{\Phi_\delta} = \left(\sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{|\widehat{f}_{\ell,k}|^2}{\widehat{\phi}_\delta(\ell)} \right)^{1/2}, \quad (15)$$

and the corresponding inner product is

$$(f, g)_{\Phi_\delta} = \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{\widehat{f}_{\ell,k} \widehat{g}_{\ell,k}}{\widehat{\phi}_\delta(\ell)}, \quad f, g \in \mathcal{N}_\Phi. \quad (16)$$

Clearly the norms $\|\cdot\|_{\Phi_\delta}$ for different δ are all equivalent, as given in the following lemma.

Lemma 2.3 (Norm-equivalence) *Let $\Pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ be a reproducing kernel of $H^\tau(\mathbb{R}^{d+1})$ with $\tau > (d+1)/2$. Let $\Phi_\delta(\mathbf{x}, \mathbf{y}) = \delta^{-d}\Pi((\mathbf{x} - \mathbf{y})/\delta)$ with $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$. Then with $\sigma = \tau - 1/2$,*

$$c_1 \|u\|_{\Phi_\delta} \leq \|u\|_{H^\sigma(\mathbb{S}^d)} \leq c_2 \delta^{-\sigma} \|u\|_{\Phi_\delta}$$

for all $u \in H^\sigma(\mathbb{S}^d)$.

Proof: See [12, Lemma 3.1]. □

From (16) it follows that the reproducing property (12) extends to general δ , that is

$$f(\mathbf{x}) = (f, \Phi_\delta(\mathbf{x}, \cdot))_{\Phi_\delta}, \quad \mathbf{x} \in \mathbb{S}^d, \quad f \in H^\sigma(\mathbb{S}^d). \quad (17)$$

3 From global to local multiscale RBF interpolation

In this section we first consider RBF interpolation with a single scale, then turn to multiscale interpolation, both global and local.

3.1 Interpolation using spherical RBFs

Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subset \Omega \subseteq \mathbb{S}^d$ be a finite set of distinct points on Ω . We define the mesh norm $h_{X,\Omega}$ and the separation radius q_X of this point set by

$$h_{X,\Omega} = \sup_{\mathbf{x} \in \Omega} \min_{\mathbf{x}_j \in X} \theta(\mathbf{x}, \mathbf{x}_j), \quad q_X = \frac{1}{2} \min_{i \neq j} \theta(\mathbf{x}_i, \mathbf{x}_j),$$

where $\theta(\mathbf{x}, \mathbf{y}) = \cos^{-1}(\mathbf{x} \cdot \mathbf{y})$ is the geodesic distance on \mathbb{S}^d . If Ω is a proper subset of \mathbb{S}^d then we say that $h_{X,\Omega}$ is a local mesh norm. If $\Omega = \mathbb{S}^d$ then the mesh norm is global, and we simply write h_X .

We define the interpolation operator $I_{X,\delta}$ associated with the set X and the kernel Φ_δ by

$$I_{X,\delta} f(\mathbf{x}) = \sum_{j=1}^N b_j \Phi_\delta(\mathbf{x}, \mathbf{x}_j), \quad I_{X,\delta} f(\mathbf{x}_j) = f(\mathbf{x}_j) \text{ for all } \mathbf{x}_j \in X. \quad (18)$$

If $\delta = 1$ then we simply write $I_X f$ instead of $I_{X,1} f$. From the interpolation condition and (17) we deduce that

$$(f - I_{X,\delta} f, \Phi_\delta(\cdot, \mathbf{x}_j))_{\Phi_\delta} = f(\mathbf{x}_j) - I_{X,\delta} f(\mathbf{x}_j) = 0, \text{ for all } \mathbf{x}_j \in X.$$

Hence $I_{X,\delta} f$ is the orthogonal projection of f into $\text{span}\{\Phi_\delta(\cdot, \mathbf{x}_j) : \mathbf{x}_j \in X\}$, from which it follows that

$$\|f - I_{X,\delta} f\|_{\Phi_\delta} \leq \|f\|_{\Phi_\delta}. \quad (19)$$

From Lemma 2.3 we then have

$$\|f - I_{X,\delta} f\|_{H^\sigma(\mathbb{S}^d)} \leq c_2 \delta^{-\sigma} \|f\|_{\Phi_\delta}. \quad (20)$$

Lemma 3.1 (Zeros Theorem) *Let $\Omega \subseteq \mathbb{S}^d$ be either an open connected region with Lipschitz boundary or $\Omega = \mathbb{S}^d$. Assume that a finite set $X \subset \Omega$ has a sufficiently small (local) mesh norm $h_{X,\Omega}$. Then, for any function $u \in H^\sigma(\Omega)$, $\sigma > d/2$, with $u|_X = 0$, for all $0 \leq \nu \leq \sigma$ we have*

$$\|u\|_{H^\nu(\Omega)} \leq Ch_{X,\Omega}^{\sigma-\nu} \|u\|_{H^\sigma(\Omega)}.$$

Proof: For $\Omega \subset \mathbb{S}^d$ an open and connected set with Lipschitz boundary, the proof follows from the zeros lemma for Lipschitz domains on a Riemannian manifold in [7, Theorem A.11]. The case $\Omega = \mathbb{S}^d$ was proved earlier in [10]. \square

Theorem 3.2 *Let $\Omega \subseteq \mathbb{S}^d$ be either a spherical cap that satisfies the conditions of Theorem 2.1 or $\Omega = \mathbb{S}^d$. Assume that a finite set $X \subset \Omega$ has a sufficiently small (local) mesh norm $h_{X,\Omega}$. Then,*

$$\|f - I_{X,\delta}f\|_{L_2(\Omega)} \leq C\delta^{-\sigma} h_{X,\Omega}^\sigma \|f\|_{H^\sigma(\Omega)}.$$

In particular, when $\delta = 1$, we have

$$\|f - I_X f\|_{L_2(\Omega)} \leq Ch_{X,\Omega}^\sigma \|f\|_{H^\sigma(\Omega)}.$$

Proof: Let $u := f - I_{X,\delta}f$, then $u|_X = 0$. Using Lemma 3.1, we have

$$\|f - I_{X,\delta}f\|_{L_2(\Omega)} \leq Ch_{X,\Omega}^\sigma \|f - I_{X,\delta}f\|_{H^\sigma(\Omega)}.$$

If Ω is a spherical cap our assumptions on Ω allow us to extend the function $f \in H^\sigma(\Omega)$ to a function $Ef \in H^\sigma(\mathbb{S}^d)$. Moreover, since $X \subset \Omega$ and $Ef|_\Omega = f|_\Omega$, the interpolant $I_{X,\delta}f$ coincides with the interpolant $I_{X,\delta}(Ef)$ on Ω . Therefore,

$$\begin{aligned} \|f - I_{X,\delta}f\|_{H^\sigma(\Omega)} &= \|Ef - I_{X,\delta}(Ef)\|_{H^\sigma(\Omega)} \leq C\|Ef - I_{X,\delta}(Ef)\|_{H^\sigma(\mathbb{S}^d)} \\ &\leq C\delta^{-\sigma} \|Ef\|_{\Phi_\delta} && \text{by (20)} \\ &\leq C\delta^{-\sigma} \|Ef\|_{H^\sigma(\mathbb{S}^d)} && \text{by Lemma 2.3} \\ &\leq C\delta^{-\sigma} \|f\|_{H^\sigma(\Omega)} && \text{by Theorem 2.1.} \end{aligned}$$

The case $\Omega = \mathbb{S}^d$ was proved in [12, Theorem 3.2]. \square

3.2 The global and local multiscale algorithm

Suppose $X_1, X_2, \dots, X_m \subset \mathbb{S}^d$ is a sequence of finite point sets with mesh norms h_1, h_2, \dots, h_m respectively. The mesh norms are assumed to satisfy $h_{j+1} \approx \mu h_j$ for some fixed $\mu \in (0, 1)$. After that, suppose $X_{m+1}, X_{m+2}, \dots, X_n \subset \Omega$ is a sequence of point sets with local mesh norms $h_{m+1,\Omega}, \dots, h_{n,\Omega}$, where $\Omega \subset \mathbb{S}^d$ is some open connected subset. In future we will write h_j for $h_{X_j,\Omega}$ for all $j = 1, \dots, n$.

Let $\delta_1, \delta_2, \dots$ be a decreasing sequence of positive real numbers defined by $\delta_j = \nu h_j$ for some $\nu > 0$. Taking the scale proportional to the mesh norm in this way is desirable for both numerical stability and efficiency, since the sparsity of the interpolation matrix

is maintained. For every $j = 1, 2, \dots$ we define the scaled SBF $\Phi_j := \Phi_{\delta_j}$, and also define the scaled approximation space $W_j = \text{span}\{\Phi_j(\cdot, \mathbf{x}) : \mathbf{x} \in X_j\}$.

We start with a widely spread set of points on the global scale and use a basis function with scale δ_1 to recover the global behavior of the function f by computing $f_1 = s_1 := I_{X_1, \delta_1} f$. The error, or residual, at the first step is $e_1 = f - f_1$. To reduce the error, at the next step we use a finer set of points X_2 and a finer scale δ_2 , and compute a correction $s_2 = I_{X_2, \delta_2} e_1$ and a new approximation $f_2 = f_1 + s_2$, so that the new residual is $e_2 = f - f_2 = e_1 - I_{X_2, \delta_2} e_1$; and so on. After m global steps we switch to local refinement, i.e. from step $(m+1)$ onwards the set X_{m+1} is localized to a small region Ω on the sphere, and the new correction s_{m+1} is constructed from the local space W_{m+1} and the new approximation is $f_{m+1} = f_m + s_{m+1}$. The multiscale algorithm then is continued for a further local $n - m$ steps.

Algorithm 1: Multiscale global/local algorithm

Data: Right hand side f , number of levels n

begin

Set $f_0 = 0, e_0 = f$.
for $j = 1, 2, \dots, n$ **do**
Determine the (global or local) interpolant $s_j \in W_j$ to e_{j-1}
Set $f_j = f_{j-1} + s_j$.
Set $e_j = e_{j-1} - s_j$.

Result: Approximation solution $f_n \in W_1 + \dots + W_n$

Remark Clearly, we could continue the algorithm by choosing an even smaller region $\Omega' \subset \Omega$ and a sequence of point sets $X_{n+1}, X_{n+2}, \dots \subset \Omega'$, and so on, until a desired resolution is reached. For simplicity of presentation, we restrict ourselves to the situation of one zooming-in region Ω in the subsequent error analysis (though not in the numerical example). Extension of the convergence theory to the general case is trivial.

We will show convergence for the scheme within a spherical cap Ω .

Theorem 3.3 (Convergence for functions in $H^\sigma(\mathbb{S}^d)$) *Let X_1, \dots, X_m be a sequence of point sets in \mathbb{S}^d and let X_{m+1}, \dots, X_n be a sequence of point sets in $\Omega \subset \mathbb{S}^d$, where Ω satisfies the requirements in Theorem 2.1. Assume that we are performing m steps of the global multilevel algorithm on \mathbb{S}^d and then $n - m$ steps of the local multilevel algorithm, localised to Ω . Let h_1, \dots, h_m be the global mesh norms and h_{m+1}, \dots, h_n be the local mesh norms of the sets X_1, \dots, X_m and X_{m+1}, \dots, X_n , respectively, and assume that, for some $\mu \in (0, 1)$, $h_{j+1} = \mu h_j$, for each $j = 1, \dots, n - 1$.*

Let Φ be a kernel generating $H^\sigma(\mathbb{S}^d)$ and let $\Phi_j := \Phi_{\delta_j}$ be defined by (13) with scale factor $\delta_j = \nu h_j$ where $1/h_1 \geq \nu \geq \gamma/\mu \geq 1$ with a fixed $\gamma > 0$. Assume that the target function f belongs to $H^\sigma(\mathbb{S}^d)$.

Then the algorithm converges in the $L_2(\Omega)$ sense linearly in the number of levels. To be more precise, there is a constant $C > 0$ and a constant $\alpha > 0$, where $\alpha < 1$ for μ sufficiently small, such that

$$\|f - f_n\|_{L_2(\Omega)} \leq C \alpha^n \|f\|_{H^\sigma(\mathbb{S}^d)}$$

for all $f \in H^\sigma(\mathbb{S}^d)$.

The theorem is a generalisation of the main result in [12].

In preparation for the proof of the theorem we first prove the following technical lemma.

Lemma 3.4 *Let e_j for $j = 0, \dots, n$ be as in Algorithm 1, and let E be the extension operator from Ω to \mathbb{S}^d as defined in Theorem 2.1. Then*

- (i) $\|e_j\|_{H^\sigma(\mathbb{S}^d)} \leq C\delta_j^{-\sigma}\|e_{j-1}\|_{\Phi_j}$ for $j = 1, \dots, m$,
- (ii) $\|e_{m+1}\|_{H^\sigma(\Omega)} \leq C\delta_{m+1}^{-\sigma}\|e_m\|_{\Phi_{m+1}}$,
- (iii) $\|e_j\|_{H^\sigma(\Omega)} \leq C\delta_j^{-\sigma}\|Ee_{j-1}\|_{\Phi_j}$ for $j = m+2, \dots, n$.

Proof: For $j = 1, \dots, m$ we have, using (20),

$$\|e_j\|_{H^\sigma(\mathbb{S}^d)} = \|e_{j-1} - I_{X_j, \delta_j} e_{j-1}\|_{H^\sigma(\mathbb{S}^d)} \leq C\delta_j^{-\sigma}\|e_{j-1}\|_{\Phi_j}.$$

For $j = m+2, \dots, n$ we have, by using the property of the extension operator and (20)

$$\begin{aligned} \|e_j\|_{H^\sigma(\Omega)} &= \|e_{j-1} - I_{X_j, \delta_j} e_{j-1}\|_{H^\sigma(\Omega)} = \|Ee_{j-1} - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\sigma(\Omega)} \\ &\leq C\|Ee_{j-1} - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\sigma(\mathbb{S}^d)} \\ &\leq C\delta_j^{-\sigma}\|Ee_{j-1}\|_{\Phi_j}. \end{aligned}$$

For the intermediate case, when $j = m+1$, we avoid the extension operator by arguing as follows

$$\begin{aligned} \|e_{m+1}\|_{H^\sigma(\Omega)} &= \|e_m - I_{X_{m+1}, \delta_{m+1}} e_m\|_{H^\sigma(\Omega)} \leq C\|e_m - I_{X_{m+1}, \delta_{m+1}} e_m\|_{H^\sigma(\mathbb{S}^d)} \\ &\leq C\delta_{m+1}^{-\sigma}\|e_m\|_{\Phi_{m+1}}. \end{aligned}$$

□

Proof of Theorem 3.3. In the proof we use repeatedly the fact that $e_j|_{X_j} = 0$, allowing us to use the zeros theorem (Lemma 3.1), and we also make essential use of the extension theorem (Theorem 2.1) and Lemma 3.4.

We start by noting that

$$\begin{aligned} \|f - f_n\|_{L_2(\Omega)} = \|e_n\|_{L_2(\Omega)} &\leq Ch_n^\sigma\|e_n\|_{H^\sigma(\Omega)} = Ch_n^\sigma\|Ee_n\|_{H^\sigma(\Omega)} \\ &\leq Ch_n^\sigma\|Ee_n\|_{H^\sigma(\mathbb{S}^d)} \\ &\leq Ch_n^\sigma\delta_{n+1}^{-\sigma}\|Ee_n\|_{\Phi_{n+1}} \\ &= C\|Ee_n\|_{\Phi_{n+1}}, \end{aligned} \tag{21}$$

where in the second-last step we use Lemma 2.3 with $\delta = \delta_{n+1}$, and in the last step $h_n/\delta_{n+1} = 1/(\mu\nu) \leq 1/\gamma$.

The result will then follow by establishing the recursions

$$\|e_j\|_{\Phi_{j+1}} \leq \alpha\|e_{j-1}\|_{\Phi_j}, \quad j = 1, \dots, m \tag{22}$$

$$\|Ee_{m+1}\|_{\Phi_{m+2}} \leq \alpha\|e_m\|_{\Phi_{m+1}}, \tag{23}$$

$$\|Ee_j\|_{\Phi_{j+1}} \leq \alpha\|Ee_{j-1}\|_{\Phi_j}, \quad j = m+2, \dots, n, \tag{24}$$

where α is some real number satisfying $0 < \alpha < 1$.

The first of these is exactly as in [12]. We shall prove the third recursion (24), noting that (22) can then be recovered by replacing Ω by \mathbb{S}^d and omitting the extension operator E .

For $j = m + 2, \dots, n$ we have by definition of the norm (15) together with (14)

$$\begin{aligned} \|Ee_j\|_{\Phi_{j+1}}^2 &= \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} \frac{|(\widehat{Ee_j})_{\ell,k}|^2}{\widehat{\phi_{\delta_{j+1}}}(\ell)} \\ &\leq C \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} |(\widehat{Ee_j})_{\ell,k}|^2 (1 + \delta_{j+1}\ell)^{2\sigma} \\ &=: I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1 &= C \sum_{\ell \leq 1/\delta_{j+1}} \sum_{k=1}^{N(d,\ell)} |(\widehat{Ee_j})_{\ell,k}|^2 (1 + \delta_{j+1}\ell)^{2\sigma} \\ &\leq C 2^{2\sigma} \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} |(\widehat{Ee_j})_{\ell,k}|^2 = C \|Ee_j\|_{L_2(\mathbb{S}^d)}^2 \\ &\leq C \|e_j\|_{L_2(\Omega)}^2 && \text{by Theorem 2.1} \\ &\leq C h_j^{2\sigma} \|e_j\|_{H^\sigma(\Omega)}^2 && \text{by Lemma 3.1} \\ &\leq C \left(\frac{h_j}{\delta_j}\right)^{2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2 && \text{by Lemma 3.4 iii)} \\ &= C \nu^{-2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2 \leq C \mu^{2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2, \end{aligned}$$

and

$$\begin{aligned} I_2 &= C \sum_{\ell > 1/\delta_{j+1}} \sum_{k=1}^{N(d,\ell)} |(\widehat{Ee_j})_{\ell,k}|^2 (1 + \delta_{j+1}\ell)^{2\sigma} \\ &\leq C (2\delta_{j+1})^{2\sigma} \sum_{\ell=0}^{\infty} \sum_{k=1}^{N(d,\ell)} |(\widehat{Ee_j})_{\ell,k}|^2 \ell^{2\sigma} \\ &\leq C \delta_{j+1}^{2\sigma} \|Ee_j\|_{H^\sigma(\mathbb{S}^d)}^2 \leq C \delta_{j+1}^{2\sigma} \|e_j\|_{H^\sigma(\Omega)}^2 && \text{by Theorem 2.1} \\ &\leq C \left(\frac{\delta_{j+1}}{\delta_j}\right)^{2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2 && \text{by Lemma 3.4 iii)} \\ &= C \mu^{2\sigma} \|Ee_{j-1}\|_{\Phi_j}^2. \end{aligned}$$

Thus we have proved $\|Ee_j\|_{\Phi_{j+1}} \leq C \mu^{-\sigma} \|Ee_{j-1}\|_{\Phi_j}$. With μ small enough, we can choose $\alpha = C \mu^\sigma < 1$, so proving the recursion (24).

Next, we discuss the switch over from global to local. We have

$$e_{m+1} = e_m - I_{X_{m+1}, \delta_{m+1}} e_m, \quad \text{with } X_m \subset \mathbb{S}^d, X_{m+1} \subset \Omega.$$

As before, we decompose

$$\|Ee_{m+1}\|_{\Phi_{m+2}}^2 = I_1 + I_2.$$

Now we have

$$\begin{aligned} I_1 &\leq C\|Ee_{m+1}\|_{L_2(\mathbb{S}^d)}^2 \\ &\leq C\|e_{m+1}\|_{L_2(\Omega)}^2 \\ &\leq Ch_{m+1}^{2\sigma}\|e_{m+1}\|_{H^\sigma(\Omega)}^2 && \text{by Lemma 3.1} \\ &\leq C\left(\frac{h_{m+1}}{\delta_{m+1}}\right)^{2\sigma}\|e_m\|_{\Phi_{m+1}}^2 && \text{by Lemma 3.4 ii).} \end{aligned}$$

The second term can be bounded by

$$\begin{aligned} I_2 &\leq C\delta_{m+2}^{2\sigma}\|Ee_{m+1}\|_{H^\sigma(\mathbb{S}^d)}^2 \leq C\delta_{m+2}^{2\sigma}\|e_{m+1}\|_{H^\sigma(\Omega)}^2 \\ &\leq C\left(\frac{\delta_{m+2}}{\delta_{m+1}}\right)^{2\sigma}\|e_m\|_{\Phi_{m+1}}^2 && \text{by Lemma 3.4 ii).} \end{aligned}$$

Hence we find

$$\|Ee_{m+1}\|_{\Phi_{m+2}}^2 \leq C\mu^{2\sigma}\|e_m\|_{\Phi_{m+1}}^2,$$

and this can be no larger than $\alpha^2\|e_m\|_{\Phi_{m+1}}^2$ for μ sufficiently small.

The first recursion (22) follows by the same proof if E is omitted and Ω is replaced by \mathbb{S}^d .

Taken in the reverse order, the recursive steps (24), (23) and (22) give

$$\begin{aligned} \|Ee_n\|_{\Phi_{n+1}} &\leq \alpha^{n-m-1}\|Ee_{m+1}\|_{\Phi_{m+2}} \\ &\leq \alpha^{n-m}\|e_m\|_{\Phi_{m+1}} \\ &\leq \alpha^n\|e_0\|_{\Phi_1} = \alpha^n\|f\|_{H^\sigma(\mathbb{S}^d)}, \end{aligned}$$

which together with (21) proves the desired result. \square

The following result on the condition numbers of the matrices is adapted from [12, Theorem 7.3].

Theorem 3.5 *Assume that the conditions in Theorem 3.3 hold, together with*

$$q_j \leq h_j \leq c_q q_j \text{ for } j = 1, 2, \dots, n \text{ with } c_q > 1.$$

There exists $C > 0$ such that the condition number of the interpolation matrices at each level of the multiscale approximation in Algorithm 1 are bounded by

$$\kappa \leq C, \quad j = 1, \dots, n.$$

4 Escaping the native space

In this section, our target function f will be assumed to be in $H^\beta(\mathbb{S}^d)$ for some $\beta \in (d/2, \sigma)$. The extension of an approximation result to spaces rougher than the native space is often referred as “escaping the native space”.

Let K be the reproducing kernel of the Sobolev space $H^{\beta+1/2}(\mathbb{R}^{d+1})$. We define the kernel Ψ by restricting K to the sphere,

$$\Psi(\mathbf{x}, \mathbf{y}) = K(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

For $0 < \delta \leq 1$, the scaled version of Ψ is defined by

$$\Psi_\delta(\mathbf{x}, \mathbf{y}) = \delta^{-d} K((\mathbf{x} - \mathbf{y})/\delta), \quad \mathbf{x}, \mathbf{y} \in \mathbb{S}^d.$$

It can be expanded into a series of spherical harmonics as

$$\Psi_\delta(\mathbf{x}, \mathbf{y}) = \sum_{\ell=0}^{\infty} \sum_{k=0}^{N(d,\ell)} \widehat{\psi}_\delta(\ell) Y_{\ell,k}(\mathbf{x}) Y_{\ell,k}(\mathbf{y}). \quad (25)$$

It is known [13, Lemma 2.1] that there are positive constants c_3, c_4 independent of δ and ℓ so that

$$c_3^2(1 + \delta\ell)^{-2\beta} \leq \widehat{\psi}_\delta(\ell) \leq c_4^2(1 + \delta\ell)^{-2\beta}, \quad \ell \geq 0. \quad (26)$$

We can define the RKHS with the reproducing kernel Ψ_δ and its norm $\|\cdot\|_{\Psi_\delta}$ as in (11) and (15). By Lemma 2.3, the norm $\|\cdot\|_{\Psi_\delta}$ defined on $\mathcal{N}_{\Psi_\delta}$ is equivalent to $\|\cdot\|_{H^\beta(\mathbb{S}^d)}$.

For the multiscale convergence theory, the sole thing that prevents us from using the proof of Theorem 3.3 with Φ_δ replaced by Ψ_δ is that a key stability property is missing: the orthogonal projection property (20) no longer holds. We therefore approximate a function in Ψ_δ by a polynomial (which of course lies in all Sobolev spaces), and apply the orthogonal projection property to that polynomial.

For a given smooth function f , the following lemma [13, Lemma 4.3] asserts the existence of a spherical polynomial that interpolates f on a set of scattered points X and, simultaneously, has a Ψ_δ norm comparable to that of f .

Lemma 4.1 *Let $f \in H^\beta(\mathbb{S}^d)$, $\beta > d/2$ and let X be a finite subset of \mathbb{S}^d with separation radius q_X . Let $\delta \in (0, 1]$ be given. There exists a constant κ , which depends only on d and β , such that if $L \geq \kappa \max\{\delta/q_X, 1/\delta\}$, then there is a spherical polynomial $p \in \mathcal{P}_L$ such that $p_X = f|_X$ and*

$$\|f - p\|_{\Psi_\delta} \leq 5\|f\|_{\Psi_\delta}.$$

Remark The dependence of the lower bound for L on the mesh radius q_X in the last lemma makes it necessary to impose a weak condition on q_X in the following theorem.

Theorem 4.2 (Convergence outside the native space) *Let X_1, \dots, X_m be a sequence of point sets in \mathbb{S}^d and let X_{m+1}, \dots, X_n be a sequence of point sets in $\Omega \subset \mathbb{S}^d$ where Ω satisfies the requirements in Theorem 2.1. Assume that we are performing m steps of the global multilevel algorithm on \mathbb{S}^d and then $n - m$ steps of the local multilevel algorithm, localised to Ω .*

Let the (global or local) mesh norms h_1, \dots, h_n and the separation radii q_1, \dots, q_n satisfy

(i) $h_{j+1} = \mu h_j$ for $j = 1, \dots, n$ with $\mu \in (0, 1)$,

(ii) $q_j \leq h_j \leq c_q \sqrt{q_j}$ for $j = 1, 2, \dots, n$.

Let Φ be a kernel generating $H^\sigma(\mathbb{S}^d)$ and let $\Phi_j := \Phi_{\delta_j}$ be defined by (13) with scale factor $\delta_j = \nu h_j$ where $1/h_1 \geq \nu \geq \gamma/\mu \geq 1$ with a fixed $\gamma > 0$. Let Ψ be a kernel generating $H^\beta(\mathbb{S}^d)$ with $\sigma > \beta > d/2$ and let $\Psi_j := \Psi_{\delta_j}$ be the scaled version (25) using the scale factor δ_j . Assume that the target function f belongs to $H^\beta(\mathbb{S}^d)$.

Then, Algorithm 1 converges in the $L_2(\Omega)$ sense linearly in the number of levels. To be more precise, there is a constant $C > 0$ and a constant $\alpha > 0$, which for μ sufficiently small is < 1 , such that

$$\|f - f_n\|_{L_2(\Omega)} \leq C\alpha^n \|f\|_{H^\beta(\mathbb{S}^d)}$$

for all $f \in H^\beta(\mathbb{S}^d)$.

Similarly to the case of Theorem 3.3, the proof of the theorem rests upon the following technical lemma. But in this case the proof is necessarily different, because the orthogonal projection property (20) is not available.

Lemma 4.3 *Let e_j for $j = 0, \dots, n$ be as in Algorithm 1, and let E be the extension operator from Ω to \mathbb{S}^d as defined in Theorem 2.1. Let the assumptions on h_j and q_j be satisfied as in Theorem 4.2. Then*

$$(i) \quad \|e_j\|_{H^\beta(\mathbb{S}^d)} \leq C\delta_j^{-\beta} \|e_{j-1}\|_{\Psi_j} \text{ for } j = 1, \dots, m,$$

$$(ii) \quad \|e_{m+1}\|_{H^\beta(\Omega)} \leq C\delta_{m+1}^{-\beta} \|e_m\|_{\Psi_{m+1}},$$

$$(iii) \quad \|e_j\|_{H^\beta(\Omega)} \leq C\delta_j^{-\beta} \|Ee_{j-1}\|_{\Psi_j} \text{ for } j = m+2, \dots, n.$$

Proof:

We prove part iii) since part i) follows easily by replacing Ω by \mathbb{S}^d and omitting the extension operator, and part (ii) is in an obvious sense intermediate. See also the proof of [13, Lemma 4.4].

We use the extension operator to extend e_{j-1} to Ee_{j-1} defined on the whole sphere, for $j = m+2, \dots, n$. Then, with $L_j := \lceil \kappa \max\{\delta_j/q_j, 1/\delta_j\} \rceil$, by Lemma 4.1, there is a polynomial $p \in \mathcal{P}_{L_j}$ that interpolates and approximates Ee_{j-1} , in the sense that

$$p|_{X_j} = Ee_{j-1}|_{X_j} \text{ and } \|p - Ee_{j-1}\|_{\Psi_j} \leq 5\|Ee_{j-1}\|_{\Psi_j}. \quad (27)$$

We note that the RBF interpolant for e_{j-1} coincides with the RBF interpolant for Ee_{j-1} on X_j . Therefore,

$$\begin{aligned} \|e_j\|_{H^\beta(\Omega)} &= \|e_{j-1} - I_{X_j, \delta_j} e_{j-1}\|_{H^\beta(\Omega)} \\ &= \|Ee_{j-1} - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\beta(\Omega)} \\ &\leq C\|Ee_{j-1} - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\beta(\mathbb{S}^d)} \\ &\leq C \left(\|Ee_{j-1} - p\|_{H^\beta(\mathbb{S}^d)} + \|p - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\beta(\mathbb{S}^d)} \right). \end{aligned} \quad (28)$$

The first term of (28) can be bounded using Lemmas 2.3 and 4.1,

$$\|Ee_{j-1} - p\|_{H^\beta(\mathbb{S}^d)} \leq c_4 \delta_j^{-\beta} \|Ee_{j-1} - p\|_{\Psi_j} \leq 5c_4 \delta_j^{-\beta} \|Ee_{j-1}\|_{\Psi_j}. \quad (29)$$

For the second term, since $p|_{X_j} = Ee_{j-1}|_{X_j}$ the interpolant $I_{X_j, \delta_j} Ee_{j-1}$ is identical to $I_{X_j, \delta_j} p$, hence by using Lemma 3.1 and (20), we have

$$\begin{aligned} \|p - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\beta(\mathbb{S}^d)} &= \|p - I_{X_j, \delta_j} p\|_{H^\beta(\mathbb{S}^d)} \leq Ch_j^{\sigma-\beta} \|p - I_{X_j, \delta_j} p\|_{H^\sigma(\mathbb{S}^d)} \\ &\leq Ch_j^{\sigma-\beta} \delta_j^{-\sigma} \|p\|_{\Phi_j} \leq C \delta_j^{-\beta} \|p\|_{\Phi_j}. \end{aligned}$$

For the polynomial p of degree L_j , using the definition (15), condition (14) and the fact that $\beta < \sigma$, we have

$$\begin{aligned} \|p\|_{\Phi_j}^2 &\leq C \sum_{\ell=0}^{L_j} \sum_{k=1}^{N(d, \ell)} (1 + \delta_j \ell)^{2\sigma} |\widehat{p}_{\ell k}|^2 \\ &\leq C(1 + \delta_j L_j)^{2(\sigma-\beta)} \sum_{\ell=0}^{L_j} \sum_{k=1}^{N(d, \ell)} (1 + \delta_j \ell)^{2\beta} |\widehat{p}_{\ell k}|^2 \\ &\leq C \|p\|_{\Psi_j}^2, \end{aligned}$$

where in the last step we used $L_j \leq C/\delta_j$. (Since $h_j \leq c_q \sqrt{q_j}$ and since $\delta_j = \nu h_j$, we see that $\delta_j/q_j \leq c/\delta_j$ and hence $L_j \leq C/\delta_j$). Thus, combining these above estimates together with the fact that $\|p\|_{\Psi_j} \leq 6\|Ee_{j-1}\|_{\Psi_j}$ we obtain

$$\|p - I_{X_j, \delta_j} Ee_{j-1}\|_{H^\beta(\mathbb{S}^d)} \leq C \delta_j^{-\beta} \|Ee_{j-1}\|_{\Psi_j}. \quad (30)$$

Combining (28), (29) and (30), we obtain the desired result. \square

Proof of Theorem 4.2. The proof is identical to that for Theorem 3.3 once we have established Lemma 4.3: the only difference is that σ is replaced by β and Φ_j by Ψ_j . We leave the details to the reader. \square

5 Numerical experiment

In this section, we describe a numerical experiment that illustrates the multiscale algorithm described in previous sections.

Let $\mathbf{p} = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$ and $\mathbf{q} = (-0.7476, 0.5069, 0.4289)^T$ be two given points on \mathbb{S}^2 , and let $\alpha := \pi/12$ and $\rho := \pi/96$. Let $\Omega_1 = G(\mathbf{q}, \alpha)$ and $\Omega_2 = G(\mathbf{q}, \rho)$ be concentric spherical caps centered at \mathbf{q} , with geodesic radii α and ρ respectively. Note that the successive areas of \mathbb{S}^2 , Ω_1 and Ω_2 are decreasing by a factor of roughly 60.

A point on \mathbb{S}^2 is parametrized by polar coordinates θ, ϕ , with

$$\mathbf{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \text{ for } \theta \in [0, \pi] \text{ and } \phi \in [0, 2\pi].$$

Let $t = \cos^{-1}(\mathbf{p} \cdot \mathbf{x})$ and let $s = \cos^{-1}(\mathbf{q} \cdot \mathbf{x})$.

The target function f is given by

$$f(\mathbf{x}) = 2 + [\sin t \cos(100t) + (1 - 3s/2\rho)_+^2 \cos(2000\theta)] S(\theta),$$

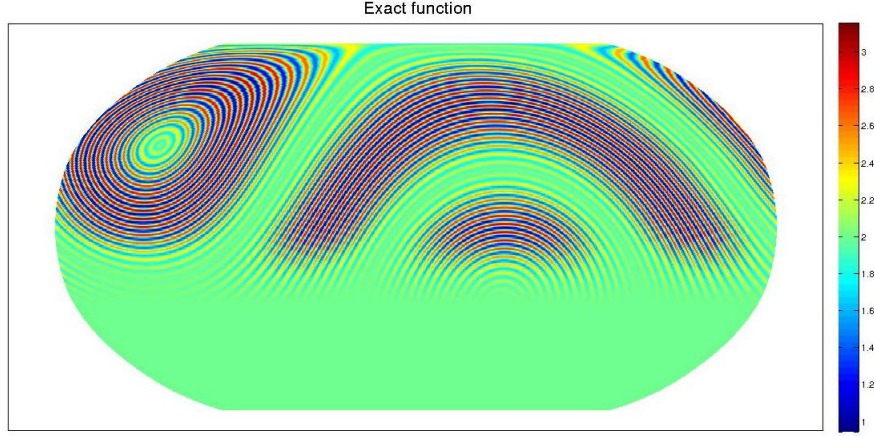


Figure 1: Exact function from a global view

where $S(\theta)$ is a cubic spline which takes the values of 1 for $\theta \in [0, \pi/2]$ and 0 for $\theta \in [2\pi/3, \pi]$. The function f , shown in Figure 1, is designed to have both global features and finer features. On the global scale, the effect of the spline multiplying the second term is that f has the constant value 2 below a latitude of 30° south. This feature was chosen because we want to be sure that the approximation scheme approximates a constant satisfactorily. (We remind the reader that approximating a constant with compactly supported radial basis functions is non-trivial, specially if the scale is comparable to the mesh norm.) The function f also contains a slow oscillation (seen in Figure 1) and a localized fast oscillation inside the spherical cap Ω_2 , as shown in left panel of Figure 5. Note that the period of the oscillation, given by the last term in the expression for f , corresponds to approximately 20 km if mapped to Earth's surface. This finer oscillation is too localized to be seen in Figure 1.

In the experiment we use 9 multiscale levels, zooming in to the cap Ω_1 after three global levels, and zooming in again to the smaller cap Ω_2 after a further three levels. In the first three (global) levels, the sets of points X_1 , X_2 , and X_3 are each centers of equal area regions generated by a partitioning algorithm [22]. The number of points in each set X_1, X_2, X_3 is increasing by a factor of 4 (see Table 1 below); the sets are not nested. The sets X_4, X_5 and X_6 are also centers of equal area regions, but the regions are partitioned from Ω_1 rather than the whole sphere. For simplicity of language we call levels 4 to 6 the “local” levels. Similarly, X_7, X_8 and X_9 are the results of partitioning Ω_2 into equal area regions. We call levels 7 to 9 the “superlocal” levels. At every stage the scale is halved exactly and the mesh norm halved approximately. The parameter details for the successive levels are given in Table 1.

The RBF used in the experiment is the Wendland function

$$\Pi(\|\mathbf{x}\|) = (1 - \|\mathbf{x}\|)_+^4(4\|\mathbf{x}\| + 1)$$

and its scaled version is

$$\Pi_\delta(\|\mathbf{x}\|) = \delta^{-2}(1 - \|\mathbf{x}\|/\delta)_+^4(4\|\mathbf{x}\|/\delta + 1),$$

Level	N	δ_j	h_j	$\ e_j\ _{L_2(\Omega_2)}$	κ_j
1	500	1/4	0.1129	4.24e-02	1.68
2	2000	1/8	0.0569	4.07e-02	1.68
3	8000	1/16	0.0281	3.45e-02	1.69
4	500	1/32	0.0186	1.56e-02	3.25
5	2000	1/64	0.0089	9.83e-03	3.39
6	8000	1/128	0.0041	8.94e-03	3.30
7	500	1/256	0.0018	7.87e-03	3.24
8	2000	1/512	0.0009	2.87e-03	3.37
9	8000	1/1024	0.0005	7.97e-04	3.28

Table 1: Parameters and local errors using 9 levels of global and local interpolation

where at level j , we set $\delta = \delta_j$. It is known that Π generates $H^3(\mathbb{R}^3)$ (see [26]) and hence the kernel $\Phi(\mathbf{x}, \mathbf{y}) = \Pi(\mathbf{x} - \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{S}^2$ generates $H^{5/2}(\mathbb{S}^2)$ (see [20]).

In Figure 2, we show the approximation after the three global levels, using the point sets X_1, X_2 and X_3 . We also show on this figure the spherical cap Ω_1 , to show the first region where we intend to zoom in. At this stage it is clear visually that the approximation scheme not yet resolved the slower oscillations, but the broad features, including the constant value in southern latitudes, are already apparent.

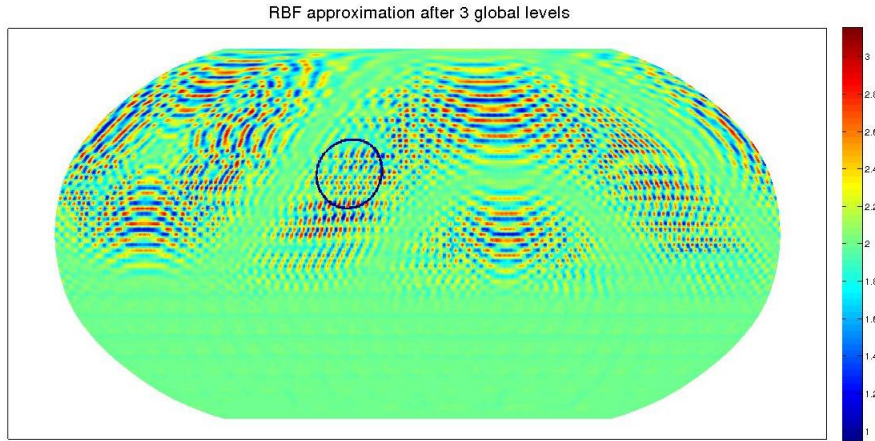


Figure 2: The global view after three global levels, with the cap Ω_1 shown

In Figure 3, we show the approximation on the spherical cap Ω_1 after 6 levels (3 global and 3 local). We also show the smaller spherical cap Ω_2 , inside which it is clear that after 6 multiscale levels the slow oscillations have largely been resolved but fine scale features have not.

Finally, in Figure 4, we show the approximation on the small spherical cap Ω_2 after 9 levels (3 global, 3 local and 3 superlocal). By this stage even the fine scale features are well resolved.

For comparison, we carry out a more modest multiscale approximation in which we

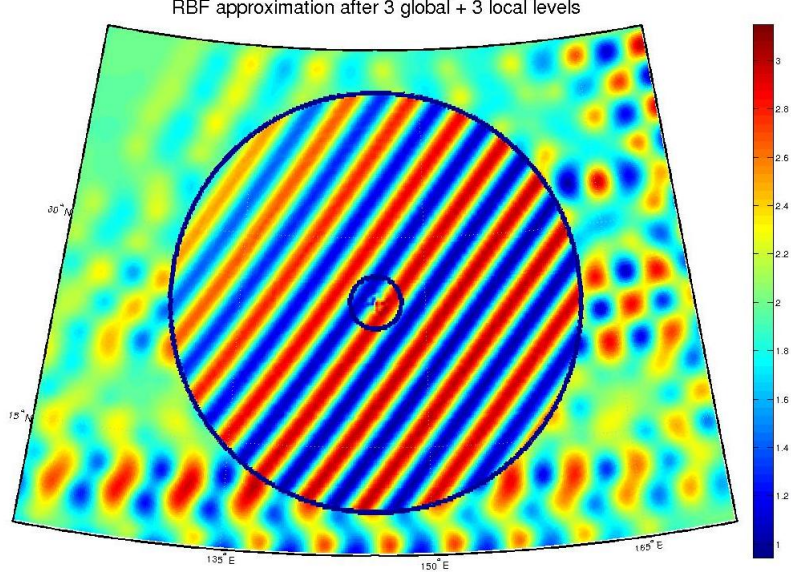


Figure 3: The local view after 3 global and 3 local levels, showing both the large cap Ω_1 and the small (“superlocal”) cap Ω_2

use just the last three (superlocal) levels, and separately also a single scale (‘one-shot’) approximation, in the second case using the final scale $\delta = 2^{-10}$ and the 8000 sampling points inside the cap Ω_2 . Poorer approximation quality of the one-shot interpolation can be seen by eye in the right panel of Figure 6. For the multiscale result in the left panel of Figure 6 that uses just the last three levels the visual result is of intermediate quality: not as good as the full multiscale result, but certainly better than the one-shot result.

Level	N	δ_j	h_j	$\ e_j\ _{L_2(\Omega_2)}$	κ_j
1	500	1/256	0.0018	2.75e-02	3.24
2	2000	1/512	0.0009	1.49e-02	3.37
3	8000	1/1024	0.0005	9.18e-03	3.28

Table 2: Local errors table when using multiscale approximation only at the last 3 superlocal levels

In Tables 1 and 2 approximate $L_2(\Omega_2)$ errors are given, in the first case for the full 9-level multiscale approximation, in the second case for the 3-level superlocal version. These were computed over a rectangular grid \mathcal{G} of size 1/64 degree times 1/64 degree restricted to the spherical cap Ω_2 ,

$$\|e_j\|_{L_2(\Omega_2)} := \left(\frac{|\Omega_2|}{|\mathcal{G} \cap \Omega_2|} \sum_{\mathbf{x}(\theta, \phi) \in \mathcal{G} \cap \Omega_2} |f(\theta, \phi) - f_j(\theta, \phi)|^2 \right)^{1/2},$$

where the area $|\Omega_2|$ of the cap Ω_2 is included so that the computed quantity is an

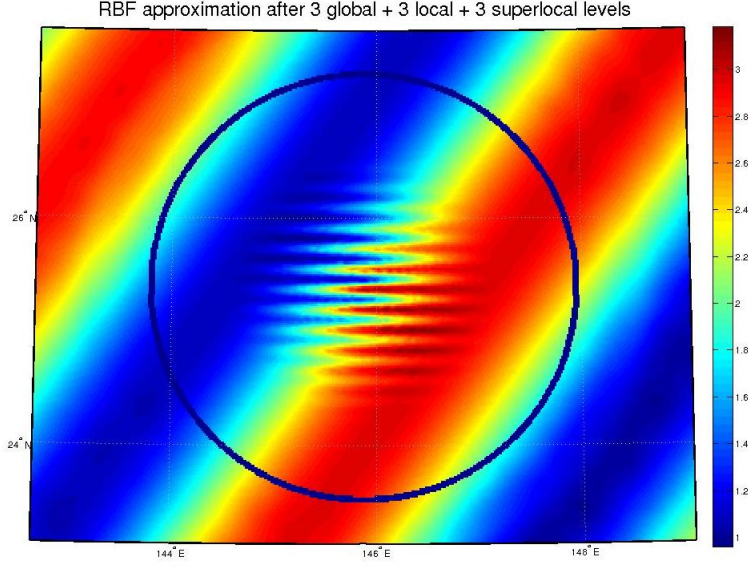


Figure 4: Superlocal view of multiscale RBF approximation after 9 levels, showing small cap Ω_2

approximation to the $L_2(\mathbb{S}^2)$ norm of the error. With the grid \mathcal{G} as above the number of points in the cap Ω_2 is $|\mathcal{G} \cap \Omega_2| = 50063$. The condition number of the interpolation matrix at level j is denoted by κ_j .

The $\|e\|_{L_2(\Omega_2)}$ error for the one-shot approximation is $2.00e - 02$, which is much larger than errors from the level 9-level multiscale approach, and also larger than the error from the 3-level multiscale approach in Table 2. Indeed, it is even an order of magnitude larger than the approximate $L_2(\Omega_2)$ norm of the function f itself, which is $7.20e - 03$. The reason for this bad result is that the one-shot approximation, with its relatively small scale compared to the mesh norm, fails to resolve well even the slowly varying background features – witness the “pepper and salt” nature of the image on the slowly varying part of the right-hand image in Figure 6. Even the 3-level multiscale approximation is struggling to resolve the slowly varying background.

A final conclusion might be that the “zooming in” multiscale approximation is successful at all levels. It could be continued indefinitely to smaller and smaller regions, giving a consistent approximation scheme at all levels if the data is available.

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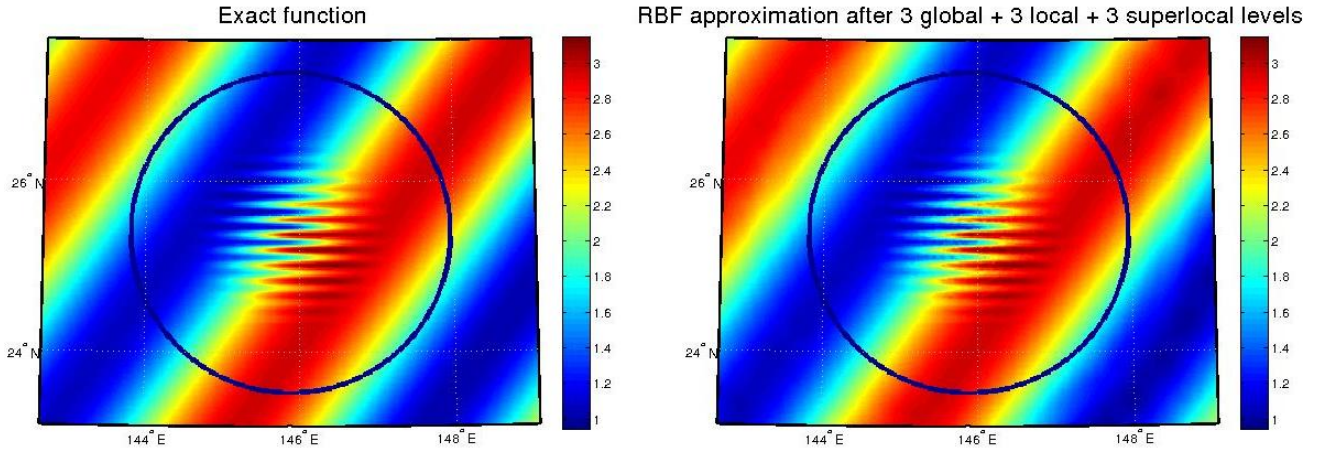


Figure 5: Exact function (left) vs. multiscale RBF approximation after 9 levels (right)

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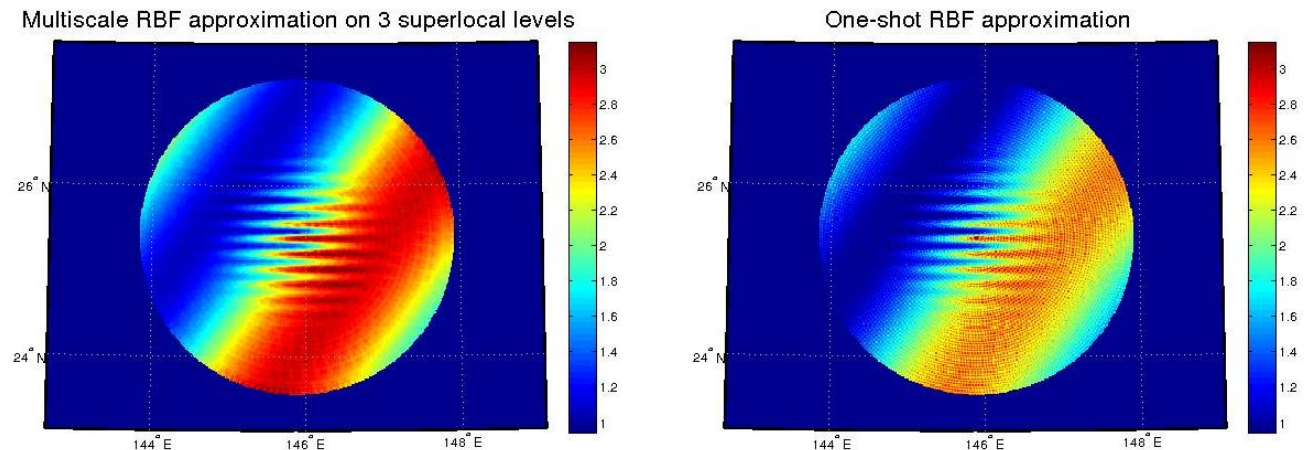


Figure 6: Multiscale RBF approximation with 3 superlocal levels (left) and one-shot interpolation on level 9 (right)

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